Descriptive set theory and geometrical paradoxes III

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The story so far

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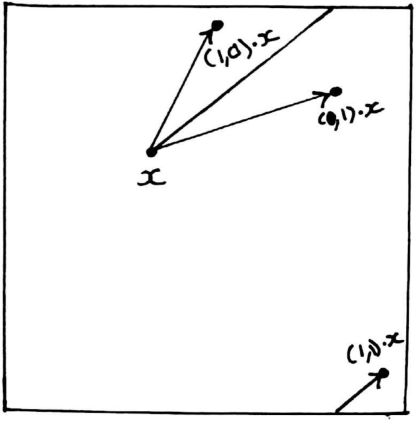
If G is a graph and f is a real-valued function on its vertices, then an f-flow of G with error ϵ is a function $\phi \colon G \to \mathbb{R}$ such that

- For every edge $(x, y) \in G$, $\phi(x, y) = -\phi(y, x)$, and
- For every vertex $x \in X$,

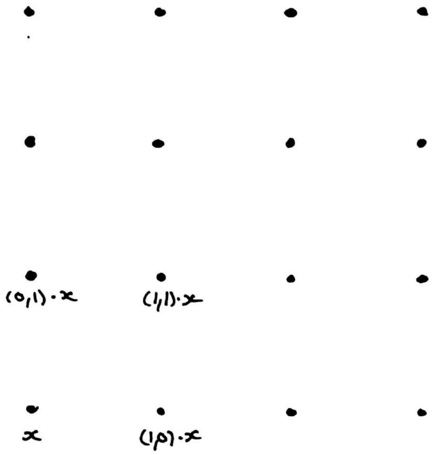
$$\left| f(x) - \sum_{(x,y) \in G} \phi(x, y) \right| < \epsilon$$

Proof overview

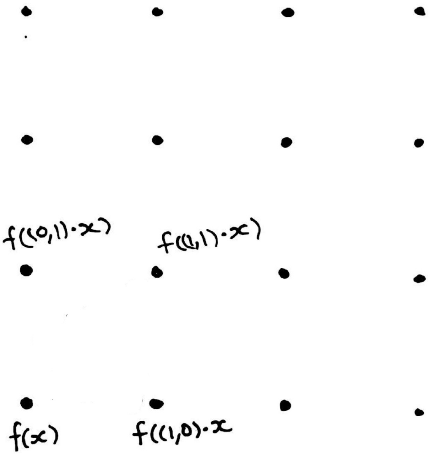
1. We construct a real-valued bounded Borel $\chi_A - \chi_B$-flow of G by giving an explicit algorithm for finding such a flow.
    - Relies on Laczkovich's discrepancy estimates.
    - Uses the fact that the average of flows is a flow.
2. We show that given any real-valued Borel $\chi_A - \chi_B$-flow of G, we can find an integer valued Borel $\chi_A - \chi_B$-flow which is "close" to the real-valued one. Uses:
    - the Ford-Fulkerson algorithm in finite combinatorics.
    - a theorem of A. Timár on boundaries of finite sets in $\mathbb{Z}^d$.
    - recent work of Gao, Jackson, Krohne and Seward on hyperfiniteness of free Borel actions of $\mathbb{Z}^d$.
3. We finish by using the proposition we proved yesterday: there's a Borel equidecomposition iff there is a bounded Borel $\chi_A - \chi_B$-flow.



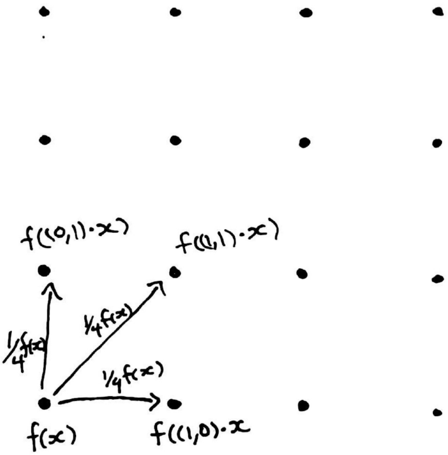
We'll describe an algorithm for constructing a real-valued f-flow where $f = \chi_A - \chi_B$ in the connected component of some $x \in \mathbb{T}^k$. We draw pictures with $k = d = 2$.



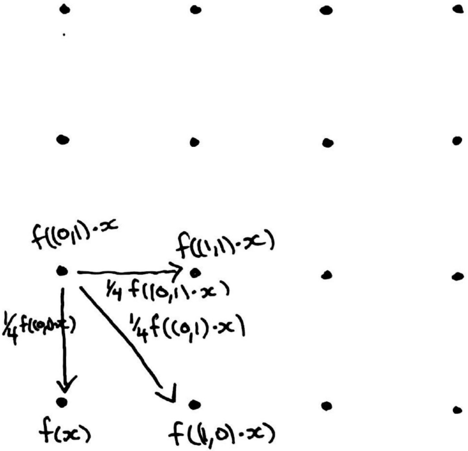
We'll draw the connected component of x in a grid (which looks like a copy of $\mathbb{Z}^d$; its orbit is infinite).



Our flow will be constructed in ω many steps. At step n we work in $2^n \times 2^n$ squares. At step 1 we consider 2×2 squares.



The idea is to spread out the error in the flow evenly over each 2×2 square. Each point contributes $1/4$ of its charge to the other 3 points.



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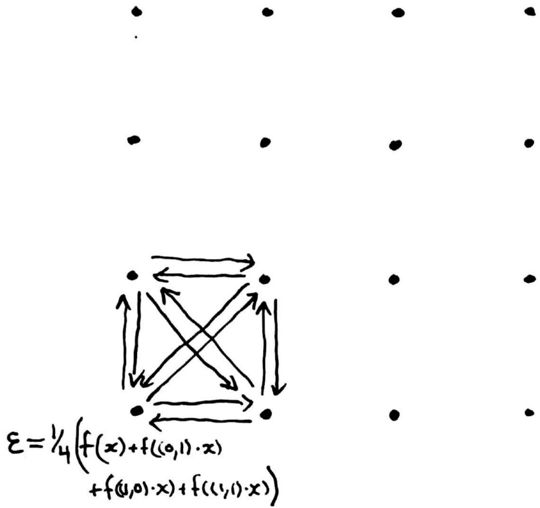
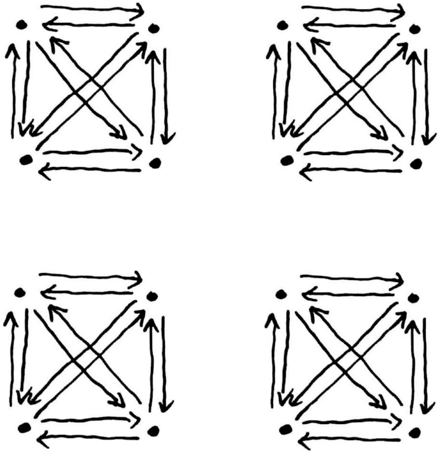
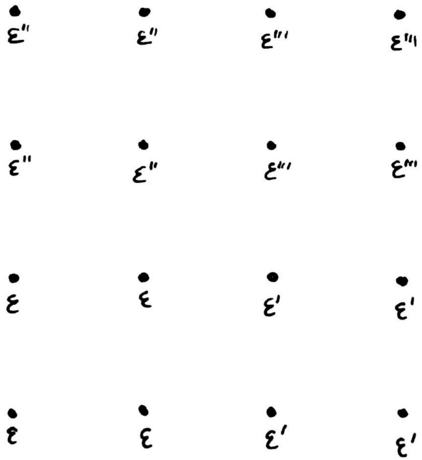


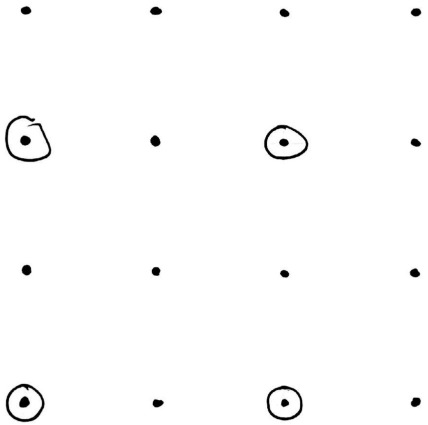
The error in the flow after step 1 is the average of f over the 2×2 square.



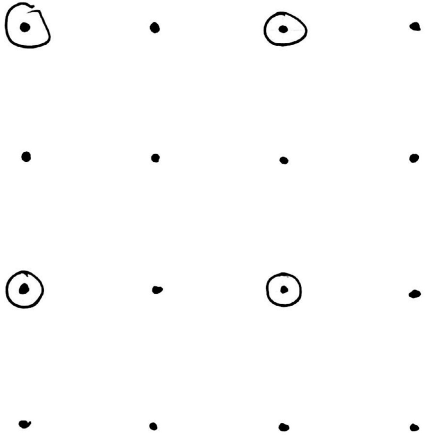
We do this for every 2×2 square in the orbit.



So the error in the flow after step 1 is the average of f on its 2×2 square.



Now we use roughly the same idea in each 4×4 square, but dealing with 4 points at a time in the way given above.



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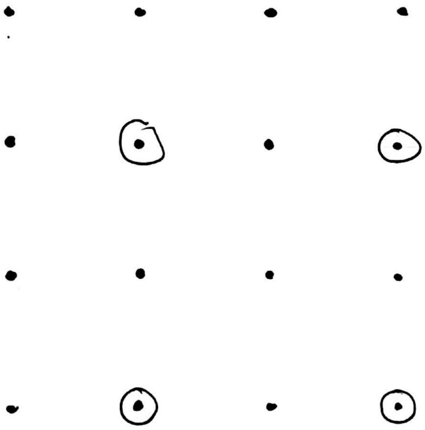

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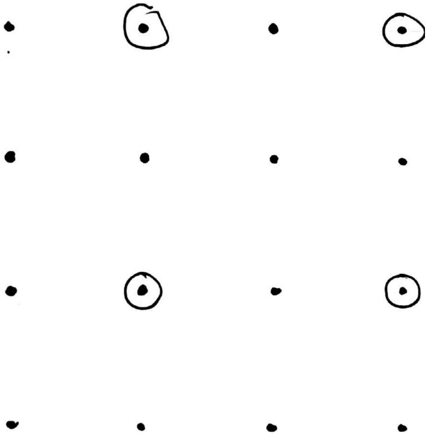

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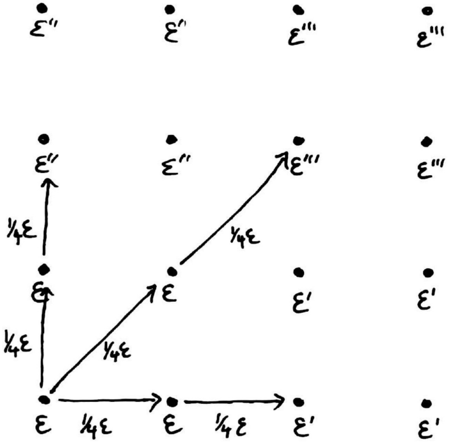
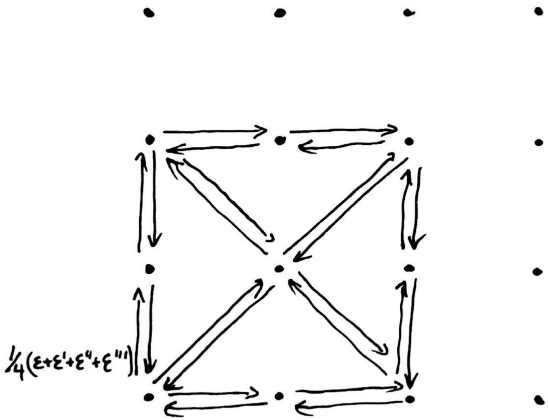
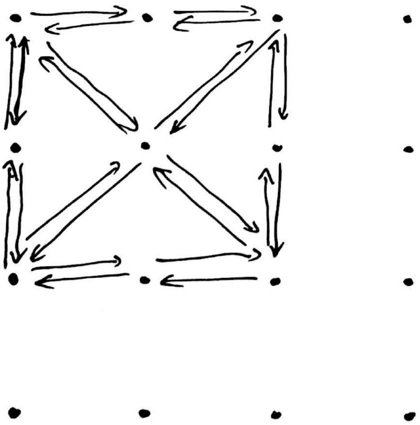


We add to the flow already constructed at the previous step.
Once again, each point contributes $1/4$ of its charge to the other 3 points.



After this second step, the error at each point will be the average of f over its 4×4 square.



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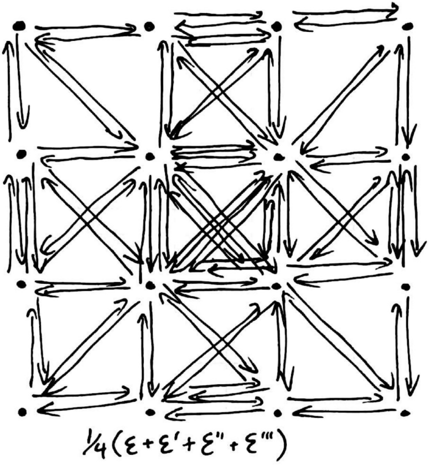

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Step 1: Constructing a real-valued flow

After step n, the error in our flow at each point will be the average value of f over the $2^n \times 2^n$ square containing the point. Since $f = \chi_A - \chi_B$, and each $2^n \times 2^n$ square contains nearly the same number of points of A and B, this error is very small.

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However, we cannot pick a single x in each orbit to be a "starting point" for this construction (since this would be a nonmeasurable Vitali set).

To fix this problem, we use an averaging trick (the average of flows is a flow!).

Step 1: Constructing a real-valued Borel flow

For every $i > 0$, let $\pi_i \colon \mathbb{Z}^d/(2^i\mathbb{Z})^d \to \mathbb{Z}^d/(2^{i-1}\mathbb{Z})^d$ be the canonical homomorphism. This yields the inverse limit

$$\hat{\mathbb{Z}}^d = \varprojlim_{i \geq 0} \mathbb{Z}^d/(2^i\mathbb{Z})^d$$

where elements of $\hat{\mathbb{Z}}^d$ are sequences $(h_0, h_1, \ldots)$ such that $\pi_i(h_i) = h_{i-1}$ for all $i > 0$. Essentially, this describes how to choose a 2×2 grid, 4×4 grid, 8×8 grid, etc. that fit inside each other.

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For each $x \in \mathbb{T}^k$ and $h \in \hat{\mathbb{Z}}^d$, our above construction yields a flow $\phi_{(x,h)}$ of the connected component of x, using the grids given by h.

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For each $x \in \mathbb{T}^k$ and $h \in \hat{\mathbb{Z}}^d$, our above construction yields a flow $\phi_{(x,h)}$ of the connected component of x, using the grids given by h. The construction is such that if $g \in \mathbb{Z}^d$, then $\phi_{(x,h)} = \phi_{(g \cdot x, -g+h)}$. Hence, the average value of this construction is invariant of our starting point ($h \mapsto -g + h$ is measure preserving):

$$\int_h \phi_{(x,h)} = \int_h \phi_{(g \cdot x, -g+h)} = \int_h \phi_{(g \cdot x, h)}$$

This average value is our real-valued Borel $\chi_A - \chi_B$ flow!

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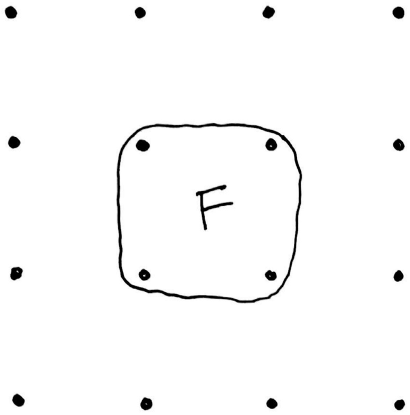
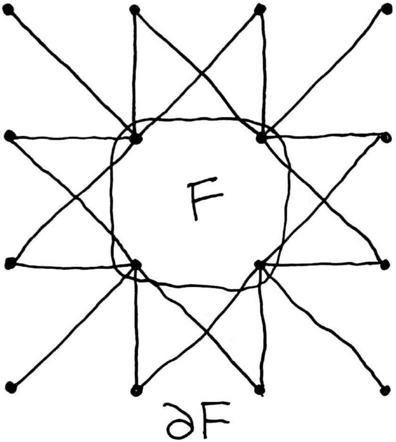
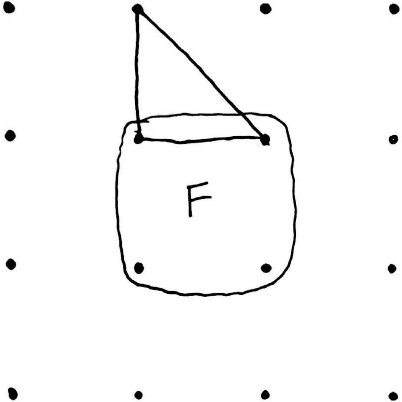
Suppose ϕ is an f-flow in G. Given a cycle in G if we add the same real value to every edge in the cycle, this preserves the property of being an f-flow. Hence, we can choose a value in $[0, 1)$ to add to this cycle so that a single edge in the cycle becomes integer.



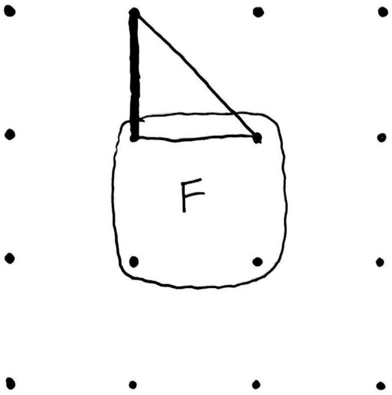
Suppose that F is a finite connected set in G.



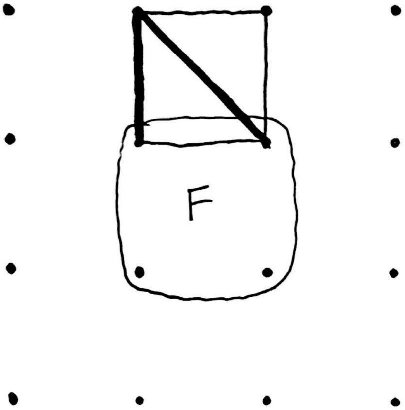
The edge boundary of F is $\partial F = \{(x, y) \in G : x \in F \wedge y \notin F\}$.
I claim we can modify the flow so that it takes integer values on ∂F.



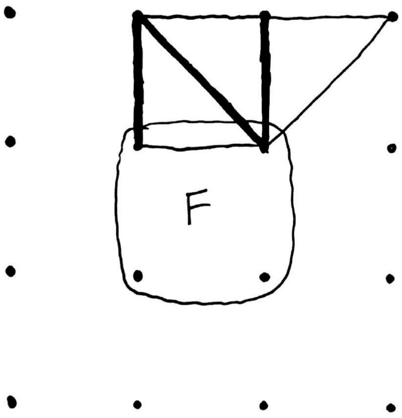
To begin, find a 3-cycle (a triangle) having an edge in ∂F.



Modify the flow on the cycle to make this edge (the darker one) integer.



Repeat this process.



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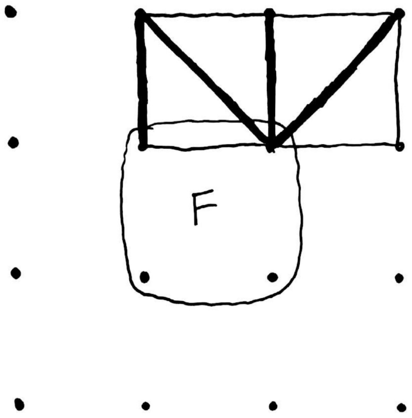

By using work of A. Timár on boundaries of finite sets in $\mathbb{Z}^d$, one can show using Euler's theorem (on the existence of Euler cycles) that for every finite set F, one can find a sequence of triangles that can be used to change the flow to be integer on ∂F.

Step 2: modifying to make an integer Borel flow

Let $[\mathbb{T}^k]^{<\infty}$ be the space of finite subsets of $\mathbb{T}^k$.

Theorem (Gao, Jackson, Krohne, and Seward, 2015)

There is a Borel set $C \subseteq [\mathbb{T}^k]^{<\infty}$ *such that*

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Use the process described on the previous slides to make the flow integer on ∂S for every $S \in C$. After removing these edges, G has finite connected components. Use the Ford-Fulkerson algorithm from finite graph theory to modify the flow on these components to be integer.

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This finishes the proof of Borel circle squaring.

Hyperfiniteness

An equivalence relation E on a Polish space X is **hyperfinite** if it is a union $E = \bigcup_i E_i$ of Borel equivalence relations $E_1 \subseteq E_2, \ldots$ on X with finite classes.

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Let Γ be a countable amenable group. Is every Borel action of Γ on a Polish space hyperfinite?

This is known to be true if we are allowed to discard a nullset by work of Ornstein and Weiss in ergodic theory (1980).

Problem 2 from the Scottish book

Open Problem (Banach, Ulam)

In every compact metric space X is there a finitely additive measure so that if A and B are Borel sets so that there is an isometry of A onto B, then A and B have equal measure?

This has a positive answer when X is countable.

The Borel Ruziewicz problem, I

Theorem (Margulis-Sullivan (1980) $n \geq 4$ Drinfeld (1984) $n = 2, 3$)

Lebesgue measure is the unique finitely additive rotation-invariant measure on the n-sphere defined on the Lebesgue measurable sets.

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Suppose $n \geq 2$. Is Lebesgue measure the unique finitely additive rotation-invariant measure on the n-sphere defined on the Borel sets?

Using the work of Drinfeld-Margulis-Sullivan, this is equivalent to asking whether every Borel Lebesgue nullset is contained in a Borel Lebesgue nullset that has a Borel paradoxical decomposition.

The Borel Ruziewicz problem, II

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Every countably additive invariant Borel probability measure on the *n*-sphere is equal to Lebesgue measure. Hence, one could equally well ask whether every finitely additive invariant Borel probability measure on the *n*-sphere is countably additive.

If this is true, it must be specific to the sphere.

Theorem (Conley, Jackson, M. Seward, Tucker-Drob, 2016)

There is a continuous free action of a nonamenable group (hence the action is paradoxical) on a Polish space so that this action admits a finitely additive invariant Borel probability measure, but does not admit any countably additive invariant Borel probability measure.

The proof uses Borel determinacy.